

Vectors

1. Basics

Vectors are powerful mathematical tools for visualizing and solving many problems in engineering and physics. The important thing to remember about a *vector quantity* is that it has both a *magnitude* and a *direction*. A *scalar quantity*, in contrast, only expresses a magnitude. As an example, consider yourself walking in a straight line. You are walking at a certain rate, but to describe your motion we also must consider the direction in which you are travelling. You are walking at a certain *speed* (e.g., 2m/s - *scalar*), but your *velocity* is the speed you are walking in a specific direction (e.g., 2 m/s, due west - *vector*).

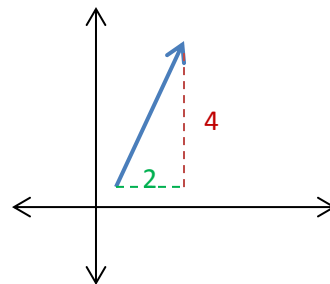


$$\vec{V} = 2 \text{ m/s, west}$$

When working with vectors, pay special attention to notation so it is clear when you are dealing with a vector and not a scalar. Vector quantities are typically denoted with **bold** type or with an arrow above, and the changes in x and y from start to finish (“tail-to-tip”) in pointy brackets:

$$\mathbf{v} = \vec{v}$$

$$\mathbf{a} = \langle 2, 4 \rangle$$



Also note that vectors can be *planar* $\langle x, y \rangle$ or *spatial* $\langle x, y, z \rangle$.

Example:

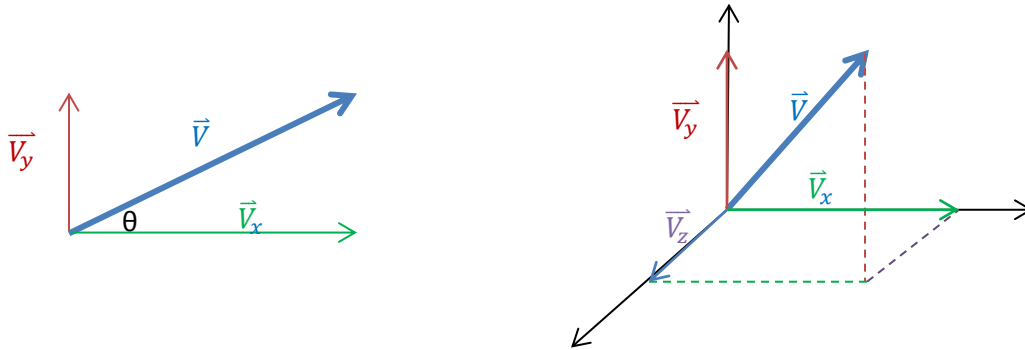
Give the vector from $(-4,3,1)$ to $(5,-2,6)$.

Solution: First, note that these are coordinate points. The vector starts at $(-4,3,1)$ (i.e., “tail”) and ends at $(5,-2,6)$ (i.e., “tip”). Find the differences in the x - and y -coordinates to get the vector:

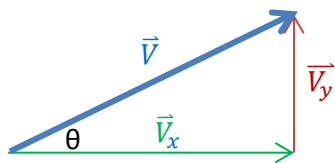
$$5 - (-4) = 9 \quad -2 - 3 = -5 \quad 6 - 1 = 5$$
$$\mathbf{a} = \langle 9, -5, 5 \rangle$$

2. Magnitude

The *magnitude* of a vector is the length of the vector. A vector is comprised of an *x-component* and a *y-component* in the 2-D plane, and *x-, y-, and z-components* in 3-D space.



We can rearrange this to form a right triangle:



and from here, use the Pythagorean Theorem to express the magnitude of a vector with the components:

$$\|\vec{v}\| = \sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2} \quad (*\text{for 2-D, } \|\vec{v}_z\| = 0)$$

For planar vectors, we can also express the components as a function of the angle, θ :

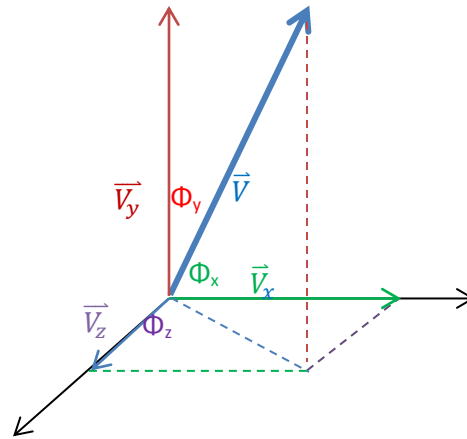
$$\|\vec{v}_x\| = \|\vec{v}\| \cos \theta \quad \|\vec{v}_y\| = \|\vec{v}\| \sin \theta$$

For spatial vectors, the components are expressed as functions of the angles between the vector and the respective axes, ϕ :

$$\cos \phi_x = \frac{\|\vec{v}_x\|}{\|\vec{v}\|} = \frac{\|\vec{v}_x\|}{\sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2}}$$

$$\cos \phi_y = \frac{\|\vec{v}_y\|}{\|\vec{v}\|} = \frac{\|\vec{v}_y\|}{\sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2}}$$

$$\cos \phi_z = \frac{\|\vec{v}_z\|}{\|\vec{v}\|} = \frac{\|\vec{v}_z\|}{\sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2}}$$



Examples:

1. Vector \vec{v} has a magnitude of 8 and acts at an angle $\theta = 35^\circ$. Find $\|\vec{v}_x\|$ and $\|\vec{v}_y\|$.

Solution: $\|\vec{v}_x\| = \|\vec{v}\| \cos \theta = 8 \cdot \cos 35^\circ = \mathbf{6.55}$

$$\|\vec{v}_y\| = \|\vec{v}\| \sin \theta = 8 \cdot \sin 35^\circ = \mathbf{4.59}$$

2. Given $\|\vec{v}_x\| = 4$, $\|\vec{v}_y\| = 10$, and $\|\vec{v}_z\| = -3$, find $\|\vec{v}\|$.

Solution:

$$\|\vec{v}\| = \sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2} = \sqrt{10^2 + 4^2 + (-3)^2} = \mathbf{11.18}$$

$$\|\vec{v}_x\| = \|\vec{v}\| \cos \theta \rightarrow \theta = \cos^{-1} \frac{\|\vec{v}_x\|}{\|\vec{v}\|} = \cos^{-1} \frac{4}{11.18}$$

$$\theta = \mathbf{85.60^\circ}$$

3. Unit Vector

Any vector having a magnitude of 1 is called a *unit vector*:

$$\|\mathbf{u}\| = \mathbf{1}$$

To find the unit vector of a vector, divide each component by the magnitude of the vector:

$$\|u_x\| = \frac{\|\vec{v}_x\|}{\|\vec{v}\|} \qquad \|u_y\| = \frac{\|\vec{v}_y\|}{\|\vec{v}\|}$$

Unit vectors in the direction of the coordinate axes are denoted by \mathbf{i} , \mathbf{j} , and \mathbf{k} , respectively. So a vector can also be expressed as scalars multiplied by each directional unit vector:

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Example:

Find a unit vector that has the same direction as $\vec{V} = \langle 3, -4 \rangle$.

Solution: First, find $\|\vec{V}\|$.

$$\|\vec{V}\| = \sqrt{\|\vec{v}_x\|^2 + \|\vec{v}_y\|^2 + \|\vec{v}_z\|^2} = \sqrt{3^2 + (-4)^2 + 0} = 5$$

Now divide each component by $\|\vec{V}\|$:

$$\mathbf{u}_V = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

4. Vector Arithmetic

Vectors can be added or subtracted either graphically or analytically. Either way, remember the following: “tip-to-tail”. This works for both addition and subtraction, but for subtraction you’ll have to “reverse” the vector that is being subtracted. Think of it like putting the vector on end and flipping it over in the opposite direction along the same line of action.

When performing arithmetical operations on vectors, the following properties apply:

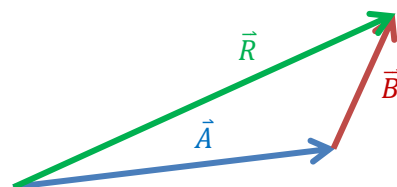
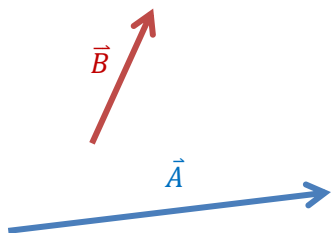
Associative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

Commutative: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

Distributive: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

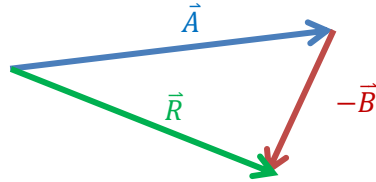
*note difference in using bold type for the vectors!

Consider these vectors:



To add these vectors, “slide” them until they are aligned “tip-to-tail”. Then draw the *resultant* vector from the tail of \vec{A} to the tip of \vec{B} . (An alternative method uses the vectors to create a parallelogram, where the diagonal is the resultant.)

To subtract these vectors, flip \vec{B} , and then add as above. (Think of it as “adding a negative.”)



The graphical method is useful for understanding how vectors work, but can be a bit cumbersome and not so accurate if you don’t get everything drawn to scale just right. Fortunately, vector arithmetic is fairly straightforward. Basically, we just add/subtract each component to get the resultant.

For: $\vec{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\vec{B} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$

$$\vec{A} \pm \vec{B} = (a \pm d)\mathbf{i} + (b \pm e)\mathbf{j} - (c \pm f)\mathbf{k}$$

Also, when multiplying (or dividing) a vector by a scalar, just multiply (or divide) each term by the scalar.

For: $\vec{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$n\vec{A} = n \cdot a\mathbf{i} + n \cdot b\mathbf{j} + n \cdot c\mathbf{k}$$

$$\frac{\vec{A}}{n} = \frac{a}{n}\mathbf{i} + \frac{b}{n}\mathbf{j} + \frac{c}{n}\mathbf{k}$$

Example:

Given the following vectors:

$$\vec{A} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$\vec{B} = 4\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$$

$$\vec{C} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$$

1. Compute $\vec{A} + \vec{B} - \vec{C}$

Solution:

$$\begin{aligned} \vec{A} + \vec{B} - \vec{C} &= (2 + 4 - 6)\mathbf{i} + (3 - 6 - 2)\mathbf{j} - (-1 - 2 + 5)\mathbf{k} \\ &= -5\mathbf{j} + 2\mathbf{k} \end{aligned}$$